CE 205: Finite Element Method: Homework I

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1. *[Geodesic problem in a plane]*: Show that a straight line has the shortest distance between any two points in a plane.

2. Hamilton's principle states that for a system of particles acted on by conservative forces, among all motions that will carry the system from a given configuration at time t_1 to a second given configuration at t_2, that which actually occurs obeys $\delta^{(1)}\mathcal{I} = 0$. Here $\mathcal{I}$ is the functional

$$\mathcal{I} = \int_{t_1}^{t_2} \mathcal{T} - [\mathcal{U} + \mathcal{V}] \ dt$$

and $\mathcal{T}$ and $\mathcal{U}$ are respectively the kinetic and potential energies of the system. $\mathcal{V}$ is the potential of the applied loads. Use Hamilton's principle to derive the equations of motion of the smoothly-sliding spring-mass system shown in Figure 1 below.

Hint: Here time is the independent variable and the displacements x_1, x_2 of the two masses the dependent variables. You will thus need two independent admissible functions η_1 and η_2 in setting up the variational scheme. Find the E-L equations of $\mathcal{I}$.

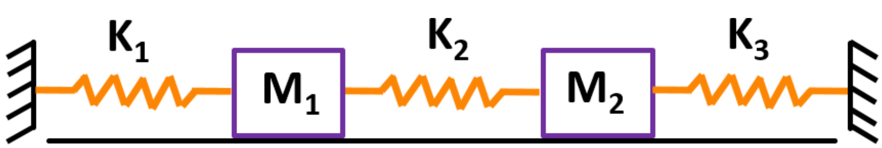

Figure 1: Spring-mass system

3. Consider the space $\mathcal{C}^1$ of first-continuously differentiable functions on an interval I. If $y \in \mathcal{C}^1$, show that $||y||_1 \equiv \max|y(x)| + \max|y'(x)|$ defines a valid norm on this vector space.

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4. Re-derive the result that the principle of virtual work

$$\int_V \rho b_i \delta u_i \, dV + \int_S t_i \delta u_i \, dS = \int_V \sigma_{ij} \delta \varepsilon_{ij} \, dV$$

is a sufficient condition for equilibrium. Proceed as done in class.

5. Consider the function $w(x)$ and the functional

$$\mathcal{I} = \int_0^L \left[\frac{EI}{2} \left(\frac{d^2 w}{dx^2} \right)^2 - q(x) w \right] dx$$

where $q(x)$ is a given function and E, I are given constants. Apply the variational operator δ formally to derive the Euler-Lagrange equation and the boundary conditions for this functional.

6. Show that the equilibrium equations are Euler-Lagrange equations for the functional $\prod$, the total potential energy of a linear elastic solid. What are the associated boundary conditions? You can assume that all applied loads are conservative.

   Hint: The solution of this problem becomes somewhat easier if you use $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$

7. Write down the total potential energy functional $\prod$ for a linear elastic rod of length L and variable stiffness $EA(x)$ acted on by a distributed load $q(x)$. Use these to find the governing differential equation of the rod and the boundary conditions.

8. Consider the following FE interpolations for rod elements with 2 d.o.f.s

$$u^{el}(x) = \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2$$
$$u^{el}(x) = \left(1 - \frac{2x}{L} + \frac{x^2}{L^2}\right) u_1 + \left(\frac{2x}{L} - \frac{x^2}{L^2}\right) u_2$$

   Do the shape functions in each case satisfy the three required properties? Explain.

9. Consider a 2-noded rod element with a tapered cross section which reduces linearly from an area $3A$ at $x = 0$ to an area A at $x = L$. Construct the element stiffness matrix. Assume the elastic modulus is E.

10. Consider a rod structure with $N = 7$ rod elements connected end-to-end. Depict the global stiffness matrix $[K]$ of this system in matrix form, highlighting the zero-entries with '0' and the non-zero entries with K_{ij} (e.g. K_{11} in position (1,1)). What is the sparsity (ratio of zero entries to total number of matrix entries) of this matrix?

    Derive an expression for the sparsity as a function of N and make a table of the sparsity values for $N = 1, 2, 3, \ldots 10$.